Problem 3.47

Supersymmetry. Consider the two operators

$$\hat{A} = i\frac{\hat{p}}{\sqrt{2m}} + W(x) \quad \text{and} \quad \hat{A}^\dagger = -i\frac{\hat{p}}{\sqrt{2m}} + W(x), \tag{3.116}$$

for some function $W(x)$. These may be multiplied in either order to construct two Hamiltonians:

$$\hat{H}_1 \equiv \hat{A}^\dagger\hat{A} = \frac{\hat{p}^2}{2m} + V_1(x) \quad \text{and} \quad \hat{H}_2 \equiv \hat{A}\hat{A}^\dagger = \frac{\hat{p}^2}{2m} + V_2(x); \tag{3.117}$$

V_1 and V_2 are called **supersymmetric partner potentials**. The energies and eigenstates of $\hat{H}_1$ and $\hat{H}_2$ are related in interesting ways.[44]

(a) Find the potentials $V_1(x)$ and $V_2(x)$, in terms of the **superpotential**, $W(x)$.

(b) Show that if $\psi_n^{(1)}$ is an eigenstate of $\hat{H}_1$ with eigenvalue $E_n^{(1)}$, then $\hat{A}\psi_n^{(1)}$ is an eigenstate of $\hat{H}_2$ with the same eigenvalue. Similarly, show that if $\psi_n^{(2)}(x)$ is an eigenstate of $\hat{H}_2$ with eigenvalue $E_n^{(2)}$, then $\hat{A}^\dagger\psi_n^{(2)}$ is an eigenstate of $\hat{H}_1$ with the same eigenvalue. The two Hamiltonians therefore have essentially identical spectra.

(c) One ordinarily chooses $W(x)$ such that the ground state of $\hat{H}_1$ satisfies

$$\hat{A}\psi_0^{(1)}(x) = 0, \tag{3.118}$$

and hence $E_0^{(1)} = 0$. Use this to find the superpotential $W(x)$, in terms of the ground state wave function, $\psi_0^{(1)}(x)$. (The fact that $\hat{A}$ annihilates $\psi_0^{(1)}$ means that $\hat{H}_2$ actually has one less eigenstate than $\hat{H}_1$, and is missing the eigenvalue $E_0^{(1)}$.)

(d) Consider the Dirac delta function well,

$$V_1(x) = \frac{m\alpha^2}{2\hbar^2} - \alpha\delta(x), \tag{3.119}$$

(the constant term, $m\alpha^2/2\hbar^2$, is included so that $E_0^{(1)} = 0$). It has a single bound state (Equation 2.132)

$$\psi_0^{(1)}(x) = \frac{\sqrt{m\alpha}}{\hbar}\exp\left[-\frac{m\alpha}{\hbar^2}|x|\right]. \tag{3.120}$$

Use the results of parts (a) and (c), and Problem 2.23(b), to determine the superpotential $W(x)$ and the partner potential $V_2(x)$. This partner potential is one that you will likely recognize, and while it has no bound states, the supersymmetry between these two systems explains the fact that their reflection and transmission coefficients are identical (see the last paragraph of Section 2.5.2).

Solution

[44] Fred Cooper, Avinash Khare, and Uday Sukhatme, *Supersymmetry in Quantum Mechanics*, World Scientific, Singapore, 2001.

Part (a)

To determine $\hat{H}_1$, let it act on a test function $f(x)$.

$$\begin{aligned}
\hat{H}_1 f(x) = \hat{A}^\dagger \hat{A} f(x) &= \left[-i\frac{\hat{p}}{\sqrt{2m}} + W(x)\right]\left[i\frac{\hat{p}}{\sqrt{2m}} + W(x)\right] f(x) \\
&= \left[\frac{\hat{p}^2}{2m} - i\frac{\hat{p}}{\sqrt{2m}}W(x) + iW(x)\frac{\hat{p}}{\sqrt{2m}} + [W(x)]^2\right] f(x) \\
&= \frac{\hat{p}^2}{2m}[f(x)] - i\frac{\hat{p}}{\sqrt{2m}}[W(x)f(x)] + iW(x)\frac{\hat{p}}{\sqrt{2m}}[f(x)] + [W(x)]^2 f(x) \\
&= \frac{\hat{p}^2}{2m}[f(x)] - \frac{i}{\sqrt{2m}}\left(-i\hbar\frac{d}{dx}\right)[W(x)f(x)] + \frac{i}{\sqrt{2m}}W(x)\left(-i\hbar\frac{d}{dx}\right)[f(x)] + [W(x)]^2 f(x) \\
&= \frac{\hat{p}^2}{2m}[f(x)] - \frac{\hbar}{\sqrt{2m}}\frac{d}{dx}[W(x)f(x)] + \frac{\hbar}{\sqrt{2m}}W(x)\frac{d}{dx}[f(x)] + [W(x)]^2 f(x) \\
&= \frac{\hat{p}^2}{2m}[f(x)] - \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx}f(x) - \cancel{\frac{\hbar}{\sqrt{2m}}W(x)\frac{df}{dx}} + \cancel{\frac{\hbar}{\sqrt{2m}}W(x)\frac{df}{dx}} + [W(x)]^2 f(x) \\
&= \left[\frac{\hat{p}^2}{2m} - \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} + [W(x)]^2\right] f(x)
\end{aligned}$$

Therefore,

$$V_1(x) = -\frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} + [W(x)]^2.$$

Find $\hat{H}_2$ in the same way.

$$\begin{aligned}
\hat{H}_2 f(x) = \hat{A}\hat{A}^\dagger f(x) &= \left[i\frac{\hat{p}}{\sqrt{2m}} + W(x)\right]\left[-i\frac{\hat{p}}{\sqrt{2m}} + W(x)\right] f(x) \\
&= \left[\frac{\hat{p}^2}{2m} + i\frac{\hat{p}}{\sqrt{2m}}W(x) - iW(x)\frac{\hat{p}}{\sqrt{2m}} + [W(x)]^2\right] f(x) \\
&= \frac{\hat{p}^2}{2m}[f(x)] + i\frac{\hat{p}}{\sqrt{2m}}[W(x)f(x)] - iW(x)\frac{\hat{p}}{\sqrt{2m}}[f(x)] + [W(x)]^2 f(x) \\
&= \frac{\hat{p}^2}{2m}[f(x)] + \frac{i}{\sqrt{2m}}\left(-i\hbar\frac{d}{dx}\right)[W(x)f(x)] - \frac{i}{\sqrt{2m}}W(x)\left(-i\hbar\frac{d}{dx}\right)[f(x)] + [W(x)]^2 f(x) \\
&= \frac{\hat{p}^2}{2m}[f(x)] + \frac{\hbar}{\sqrt{2m}}\frac{d}{dx}[W(x)f(x)] - \frac{\hbar}{\sqrt{2m}}W(x)\frac{d}{dx}[f(x)] + [W(x)]^2 f(x) \\
&= \frac{\hat{p}^2}{2m}[f(x)] + \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx}f(x) + \cancel{\frac{\hbar}{\sqrt{2m}}W(x)\frac{df}{dx}} - \cancel{\frac{\hbar}{\sqrt{2m}}W(x)\frac{df}{dx}} + [W(x)]^2 f(x) \\
&= \left[\frac{\hat{p}^2}{2m} + \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} + [W(x)]^2\right] f(x)
\end{aligned}$$

Therefore,

$$V_2(x) = \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} + [W(x)]^2.$$

Part (b)

Suppose that $\psi_n^{(1)}$ is an eigenstate of $\hat{H}_1$ with eigenvalue $E_n^{(1)}$.

$$\hat{H}_1\psi_n^{(1)} = E_n^{(1)}\psi_n^{(1)}$$

Show that $\hat{A}\psi_n^{(1)}$ is an eigenstate of $\hat{H}_2$ with eigenvalue $E_n^{(1)}$.

$$\begin{aligned}
\hat{H}_2\left[\hat{A}\psi_n^{(1)}\right] &= \hat{A}\hat{A}^\dagger\left[\hat{A}\psi_n^{(1)}\right] \\
&= \hat{A}\left[\hat{A}^\dagger\hat{A}\psi_n^{(1)}\right] \\
&= \hat{A}\left[\hat{H}_1\psi_n^{(1)}\right] \\
&= \hat{A}\left[E_n^{(1)}\psi_n^{(1)}\right] \\
&= E_n^{(1)}\left[\hat{A}\psi_n^{(1)}\right]
\end{aligned}$$

Now suppose that $\psi_n^{(2)}$ is an eigenstate of $\hat{H}_2$ with eigenvalue $E_n^{(2)}$.

$$\hat{H}_2\psi_n^{(2)} = E_n^{(2)}\psi_n^{(2)}$$

Show that $\hat{A}^\dagger\psi_n^{(2)}$ is an eigenstate of $\hat{H}_1$ with eigenvalue $E_n^{(2)}$.

$$\begin{aligned}
\hat{H}_1\left[\hat{A}^\dagger\psi_n^{(2)}\right] &= \hat{A}^\dagger\hat{A}\left[\hat{A}^\dagger\psi_n^{(2)}\right] \\
&= \hat{A}^\dagger\left[\hat{A}\hat{A}^\dagger\psi_n^{(2)}\right] \\
&= \hat{A}^\dagger\left[\hat{H}_2\psi_n^{(2)}\right] \\
&= \hat{A}^\dagger\left[E_n^{(2)}\psi_n^{(2)}\right] \\
&= E_n^{(2)}\left[\hat{A}^\dagger\psi_n^{(2)}\right]
\end{aligned}$$

Part (c)

Choose the superpotential $W(x)$ so that

$$\begin{aligned}
0 &= \hat{A}\psi_0^{(1)}(x) \\
&= \left[i\frac{\hat{p}}{\sqrt{2m}} + W(x)\right]\psi_0^{(1)}(x) \\
&= \left[\frac{i}{\sqrt{2m}}\left(-i\hbar\frac{d}{dx}\right) + W(x)\right]\psi_0^{(1)}(x) \\
&= \left[\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right]\psi_0^{(1)}(x) \\
&= \frac{\hbar}{\sqrt{2m}}\frac{d\psi_0^{(1)}}{dx} + W(x)\psi_0^{(1)}(x).
\end{aligned}$$

Solve for $W(x)$.

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\frac{d\psi_0^{(1)}}{dx}}{\psi_0^{(1)}(x)}$$

Therefore,

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \ln \left| \psi_0^{(1)}(x) \right| .$$

Part (d)

For the given bound state,

$$\psi_0^{(1)}(x) = \frac{\sqrt{m\alpha}}{\hbar} \exp\left(-\frac{m\alpha}{\hbar^2}|x|\right),$$

which is always positive, the superpotential is

$$\begin{aligned}
W(x) &= -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \ln \psi_0^{(1)}(x) \\
&= -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \ln \left[\frac{\sqrt{m\alpha}}{\hbar} \exp\left(-\frac{m\alpha}{\hbar^2}|x|\right) \right] \\
&= -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \left[\ln \frac{\sqrt{m\alpha}}{\hbar} + \ln \exp\left(-\frac{m\alpha}{\hbar^2}|x|\right) \right] \\
&= -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \left[\ln \frac{\sqrt{m\alpha}}{\hbar} + \left(-\frac{m\alpha}{\hbar^2}|x|\right) \right] \\
&= -\frac{\hbar}{\sqrt{2m}} \left[-\frac{m\alpha}{\hbar^2} \left(\frac{d}{dx}|x| \right) \right] \\
&= \frac{\alpha}{\hbar}\sqrt{\frac{m}{2}} \frac{d}{dx}\left(\sqrt{x^2}\right) \\
&= \frac{\alpha}{\hbar}\sqrt{\frac{m}{2}} \left[\frac{1}{2}\left(x^2\right)^{-1/2} \cdot \frac{d}{dx}(x^2) \right] \\
&= \frac{\alpha}{\hbar}\sqrt{\frac{m}{2}} \left[\frac{1}{2\sqrt{x^2}} \cdot (2x) \right] \\
&= \frac{\alpha}{\hbar}\sqrt{\frac{m}{2}} \left(\frac{x}{|x|} \right) \\
&= \frac{\alpha}{\hbar}\sqrt{\frac{m}{2}} \operatorname{sgn} x \\
&= \frac{\alpha}{\hbar}\sqrt{\frac{m}{2}} \left[\theta(x) - \theta(-x)\right],
\end{aligned}$$

so the corresponding supersymmetric partner potential is

$$\begin{aligned}
V_2(x) &= \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} + [W(x)]^2 \\
&= \frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\left\{\frac{\alpha}{\hbar}\sqrt{\frac{m}{2}}\,[\theta(x) - \theta(-x)]\right\} + \left(\frac{\alpha}{\hbar}\sqrt{\frac{m}{2}}\operatorname{sgn} x\right)^2 \\
&= \frac{\hbar}{\sqrt{2m}}\frac{\alpha}{\hbar}\sqrt{\frac{m}{2}}\left[\frac{d}{dx}\theta(x) - \frac{d}{dx}\theta(-x)\right] + \frac{\alpha^2}{\hbar^2}\frac{m}{2} \\
&= \frac{\alpha}{2}\left[\theta'(x) - \theta'(-x)\cdot\frac{d}{dx}(-x)\right] + \frac{\alpha^2}{\hbar^2}\frac{m}{2} \\
&= \frac{\alpha}{2}[\delta(x) - \delta(-x)\cdot(-1)] + \frac{\alpha^2}{\hbar^2}\frac{m}{2} \\
&= \frac{\alpha}{2}[\delta(x) + \delta(-x)] + \frac{\alpha^2}{\hbar^2}\frac{m}{2} \\
&= \frac{\alpha}{2}\left[\delta(x) + \frac{1}{|-1|}\delta(x)\right] + \frac{\alpha^2}{\hbar^2}\frac{m}{2} \\
&= \frac{\alpha}{2}\,[\delta(x) + \delta(x)] + \frac{\alpha^2}{\hbar^2}\frac{m}{2} \\
&= \frac{m\alpha^2}{2\hbar^2} + \alpha\delta(x).
\end{aligned}$$

This is a delta-function barrier. See the bottom of page 68.